Title: Iterating the Function $f(z) = z^2 + c$ over the Complex Numbers

Brief Overview:

Students will use a spreadsheet to study the sequences of complex numbers obtained by iterating the function $f(z) = z^2 + c$ where z and c are complex numbers. They will discover the various types of behavior that can result--divergence, convergence to fixed points and cycles, and chaos, depending upon the value of c.

Links to NCTM Standards:

- **Mathematics as Communication**
  Students will explain their findings in writing.

- **Mathematics as Reasoning**
  Students will use inductive reasoning to discover how the value of c affects the behavior of $f(z) = z^2 + c$ under iteration. Then they will use deductive reasoning to prove some of their findings.

- **Mathematical Connections**
  This topic makes connections among the following branches of mathematics: algebra, discrete mathematics, and analysis.

- **Algebra**
  Students will solve quadratic equations to find and prove the existence of fixed points and attractors.

- **Functions**
  Students will study functions from the perspective of iteration.

- **Discrete Mathematics**
  Students will study the long term behavior of sequences resulting from iterating the function $f(z) = z^2 + c$.

- **Conceptual Underpinnings of Calculus**
  Students will study limits of sequences obtained by iteration.

Grade/Level:

Grades 10 - 12

Duration/Length:

3 - 4 days, depending on students' prior knowledge

Prerequisite Knowledge:

Students should have working knowledge of the following concepts:

- Functions and function notation
- Iteration of functions

- Complex numbers: operations and graphing
- Sequences and limits

Objectives:

Students will:

- use a spreadsheet to explore the orbits of the function $f(z) = z^2 + c$.
- use algebra to explain their findings.
- find organization to the behavior of orbits as c varies.

Materials/Resources/Printed Materials:

- Calculators
- Graph paper
- Spreadsheet software

Development/Procedures:

Prepare a spreadsheet file ahead of time as follows:

	A	B	C
1		real	imaginary
2	c =		
3			
4	Iteration	Orbit	
5	0	0	0
6	=a5+1	=b5*b5-c5*c5+b$2	=2*b5*c5+c$2

Copy row 6 to rows 7 through 105. This will calculate the first 100 iterations of the function $f(z) = z^2 + c$, where the value of c is entered by the user in cells B2 (real part) and C2 (imaginary part), and the initial value z_0 is entered in cells B5 (real part) and C5 (imaginary part). Create an xy- or scatter-plot of these iterates in the complex plane by setting the x-series to cells B5 through B105 and the y-series to cells C5 through C105. The specific commands to do this vary depending on the software used, so check your documentation if necessary. Set the x-axis and y-axis limits to $-2 < x < 2$. Position the graph so it is visible beside columns A through C. Make this spreadsheet available on each computer your students will use.

Begin class by reviewing the concepts of function, iteration, and complex numbers. Review notation such as $z_1 = f(z_0)$, $z_2 = f(z_1)$, …, $z_{n+1} = f(z_n)$, as well as vocabulary such as initial point, orbit, attracting fixed point, and repelling fixed point.

Now tell students that they will study iteration of functions of the form $f(z) = z^2 + c$, where z and c are complex numbers. They will start with $c = 0$. Have them find the first few iterates for various values of z_0 by hand with the aid of a calculator. Then have students graph them in the complex plane by hand. Use some values of z_0 which are less than 1 in absolute value, and some which are greater than 1 in absolute value.

For example: If $z_0 = .5 + .2i$, then $z_1 = .21 + .2i$, $z_2 = .0041 + .084i$, and $z_3 = -.007 + .00069i$. The sequence tends toward 0. If $z_0 = 1 + 2i$, then $z_1 = -3 + 4i$, $z_2 = -7 - 24i$, and $z_3 = -527 + 336i$. This sequence diverges.

Then have students find iterates of the function $f(z) = z^2 + c$ when c is not 0. For example, let $c = -.5 + 25i$. Now if $z_0 = 2 + i$, then $z_1 = 2.5 + 4.25i$, and $z_2 = -12.3 + 21.5i$. This sequence diverges.

Now have students repeat the above work on the spreadsheet. First, for $c = 0$, have them type 0 in cells B2 and C2. Then in cells B5 and C5 they should type the real and imaginary parts of z_0. They should retry the value $z_0 = .5 + .2i$ as they did by hand to confirm their results. They should see a graph of points that tend toward the origin. When they retry $z_0 = 1 + 2i$ they will see only a couple of points because the rest are off the screen.

Have students work in pairs to experiment with other values of z_0, and together come up with a conjecture. They should notice that when $|z_0| < 1$, iterates converge to 0. When $|z_0| > 1$, iterates diverge. Have them try to explain why this is true. They will need to use the fact that for any complex number z, $|z^2| = |z|^2$, and for any real number x with $|x| < 1$, the limit of x^n as n approaches infinity is 0. On the other hand, when $|x| > 1$, as n approaches infinity, x^n diverges. Thus, 0 is an attractor, and note that it is a fixed point; that is, $f(0) = 0$. Have them find another fixed point by solving $f(x) = x$. They should discover that 1 is a fixed point, but it is repelling.

Define the basin of attraction for an attractor as the set of all initial points whose orbits become arbitrarily close to the attractor. Thus, the basin of attraction for the attractor 0 (using $f(z) = z^2$) is the set of all z_0 with $|z_0| < 1$.

Ask students what happens when $|z_0| = 1$. For example, they can try $z_0 = .8 + .6i$. In this case, iterates theoretically remain on the circle $|z| = 1$. However, because of limits to the computer's precision, the orbit will eventually converge or diverge. But until it does, note that the orbit traces out the edge of the basin of attraction. This is an important idea that will come up again.

Now have students repeat this process with $c = -.5 + .25i$ and confirm their earlier result with $z_0 = 2 + i$. Have them try to find orbits which converge to an attracting fixed point. For example, when $z_0 = .1 + .2i$, the orbit appears to converge to approximately $-.378 + .142i$. Thus, $z_0 = .1 + .2i$ is inside the basin of attraction. But when $z_0 = .1 + .8i$, the orbit diverges. Have students try to find a point right on the boundary of the basin by varying only the imaginary part. The author found that $z_0 = .1 + .76760940922i$ is close to the boundary and traces out an interesting boundary before being attracted to the fixed point. (Actually, this boundary is known as a Julia set, and is a fractal.)

Now have students try $c = -.8 + .1i$, and initial point $z_0 = 0$. This time, iterates clump around two points. If they examine columns B and C farther down in the spreadsheet, they will notice that the iterates are alternating between two values, $-.216 - .176i$ and $-.784 +.176i$. This is an attracting 2-cycle.

Much other interesting behavior can be found by experimentation. Have students discover it. For example, 3-cycles can be found for $c = -.1+.7i$. Gradually increase the imaginary part of this and watch the 3-cycle split into a 6-cycle for $c = -.1 + .87i$ and then explode for larger imaginary parts. It is convenient to leave $z_0 = 0$ throughout, since it has been proven that 0 is always in the basin of attraction when one exists for $f(z) = z^2 + c$. Another interesting value to try is $c = -.8 + .15$. In this case, orbits will trace out two doughnut-shaped patterns in the complex plane. If the actual iterates in columns B and C are observed, the numbers appear to be random and unpredictable. This is chaos.

An infinite variety of behavior can be found, just by varying the value of c. Have students explore various values, and record on the chalkboard those that they find interesting so that other students can try them. Various spiral patterns and cycles of many different periods can be found. They may discover that as c varies continuously, cycles split (for example, from a 5-cycle to a 10-cycle or 15-cycle), then result in chaotic doughnuts, then explode into diverging orbits. This part of the investigation is very enjoyable and some students (as well as the author) could spend hours on it. Some good values of c to try are:

$c = .296875 - .03125i$: spiral into an attracting fixed point
$c = -.546875 + .453125i$: spiral into a cycle
$c = -.453125 - .546875i$: doughnut shape
$c = -.484375 - .515625i$: spiral into a 5-cycle
$c = -.67809 - .273925i$: spiral into a cycle
$c = -.699 + .125i$: hurricane

Performance Assessment:

Have students write a paper that describes their investigation and what they found. They should describe the role of c and z_0 in determining the sequence of iterates. They should explain what an orbit is and how it can be graphed, as well as the kinds of behavior that can result. They should define the term basin of attraction, and describe how it can be found roughly. They should attempt rough sketches of some of the basins they found. They should describe how the behavior of orbits changes as the value of c changes.

A holistic scoring rubric can be used to evaluate this work in five areas:

(1) Description of mathematical concepts involved (functions, iteration, complex numbers, graphing, orbits, and limits)
(2) Explanation of procedures used (use of spreadsheet, varying c and z_o)
(3) Description of findings (behaviors such as divergence, convergence to a limit, cycles, and chaos; and how they depended on c and z_o)
(4) Use of notation and algebraic manipulation
(5) Use of logical reasoning

Each of these areas can be scored on the following scale:

4	Work is correct and complete.
3	Work is almost correct and complete; some errors made or details omitted.
2	Work shows a general understanding on the part of the student, but notable gaps or errors are present.
1	Some work is correct, showing minimal incomplete understanding, but there is little or no chain of reasoning.
0	Work is wrong or meaningless. There is no evidence of understanding.

Extension/Follow Up:

Students could write their own computer program to generate clearer pictures of the basins of attraction, or Julia sets. For information on how to do that, they can refer to Robert Devaney's book "Chaos, Fractals, and Dynamics: Computer Experiments in Mathematics" (1990, Addison-Wesley), chapters 5 and 6. To classify the different kinds of Julia sets and find some order in how they change as c changes, they can learn about the Mandelbrot set in chapter 8 of the same book.

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